

# No-Go Theorem for Boson Condensation in Topologically Ordered Quantum Liquids

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Certain phase transitions between topological quantum field theories (TQFT) are driven by the condensation of bosonic anyons. However, as bosons in a TQFT are themselves nontrivial collective excitations, there can be topological obstructions that prevent them from condensing. Here we formulate such an obstruction in the form of a no-go theorem. We use it to show that no condensation is possible in  $SO(3)_k$  TQFTs with odd  $k$ . We further show that a “layered” theory obtained by tensoring  $SO(3)_k$  TQFT with itself any integer number of times does not admit condensation transitions either. This includes (as the case  $k = 3$ ) the noncondensability of any number of layers of the Fibonacci TQFT.

Topological order, a fundamental concept in quantum many-body physics, is best understood in two-dimensional gapped quantum liquids, such as the fractional quantum Hall effect and certain spin liquids [1–9]. In these systems, quasi-particle excitations with anyonic quantum-statistical properties emerge [10]. Their fusion and braiding behavior at large distances define a topological quantum field theory (TQFT), which characterizes the universal properties of the phase [11–14].

The phase transitions between topological phases are, most of the times, driven by the condensation of bosons [11, 15–25]. In the context of TQFTs, a boson is an emergent quasi-particle in the topologically ordered phase with bosonic self-statistics, but which could have nontrivial fusion and braiding relations with the other anyons. Such a quasiparticle can potentially undergo Bose-Einstein condensation, causing a phase transition to another topologically ordered phase. The topological data of the new phase can be inferred from those of the initial topological order [25].

One motivation to study condensation transitions is to classify topological order. An important example are the 16 types of gauged chiral superconductors introduced by Kitaev [3]. Kitaev showed that while two-dimensional superconductors are classified by an integer  $\mathbb{Z}$ , only 16 bulk phases are topologically distinct. This construction can be understood by considering several layers of the elementary (Ising) TQFT. Coupling the layers by condensing inter-layer cooper pairs, one obtains exactly 16 distinct TQFTs including Ising, the toric code and the double semion model. They determine the nature of the topologically protected excitations in the vortices of each superconductor, including their braiding statistics. In essence,  $\mathbb{Z}_{16}$  classification is a property of the Ising TQFT.

It is imperative to ask whether multi-layer systems of other TQFTs show a similar collapse of the classification from  $\mathbb{Z}$  to  $\mathbb{Z}_N$  for some integer  $N$ . In this paper, we derive a criterion for when this is not the case, i.e., when the  $\mathbb{Z}$  classification generated by a given TQFT is stable. This criterion is based on the fact that there exist bosonic anyons that cannot be condensed. An example are the bosons in multi-layered Fibonacci topological order [17, 25, 26]. In this work, we gen-

eralize this observation by formulating a no-go theorem that constitutes a sufficient obstruction against the condensation of a boson. Our criterion and its proof are given using the tensor category formulation of topological order [3, 27–34], which we can use to describe the condensation transition axiomatically [16, 17, 25]. We apply our no-go theorem to several examples, including the forementioned multi-layer Fibonacci TQFTs.

*Formalism* — We use the algebraic formulation of anyon condensation discussed in Ref. 25. Here we simply restate the important relations and refer the reader to Ref. 25 for details. A *fusion category* is characterized by a set of anyons  $a, b, c, \dots$  and fusion rules  $a \times b = \sum_c N_{ab}^c c$  between them. The quantum dimension  $d_a$  gives the size of the nonlocal internal Hilbert space associated with anyon  $a$ , and is equal to largest eigenvalue of the matrix  $N_a$  with elements  $(N_a)_{bc} \equiv N_{ab}^c$ . A *braided tensor category* has additional structure, of which we will use the topological spin  $\theta_a$  of  $a$ , a complex number with  $|\theta_a| = 1$ . Bosons are defined by  $\theta_a = 1$ . A special role is played by the vacuum anyon as the unique identity element of fusion. It is a boson with quantum dimension 1.

Condensation is based on a mapping, called restriction, between the anyons  $a$  in the original TQFT  $\mathcal{A}$  and the anyons  $t$  in the condensed fusion category  $\mathcal{T}$  characterized by integers  $n_a^t \in \mathbb{Z}_{\geq 0}$ :

$$a \mapsto a^\downarrow \equiv \sum_{t \in \mathcal{T}} n_a^t t, \quad \forall a \in \mathcal{A}. \quad (1)$$

If more than one particle appears on the right-hand side of Eq. (1), we say that the  $a$  particle *splits*. If  $n_a^t \neq 0$ , we say  $t$  is in the *restriction* of  $a$  or  $t \in a^\downarrow$ . We require that  $n_1^\varphi = 1$ , where  $\varphi$  and 1 are the vacua in  $\mathcal{T}$  and  $\mathcal{A}$ , respectively. Imposing that condensation commutes with fusion implies the fundamental relation [25]

$$\sum_{r,s \in \mathcal{T}} n_a^r n_b^s \tilde{N}_{rs}^t = \sum_{c \in \mathcal{A}} N_{ab}^c n_c^t, \quad (2)$$

between the fusion coefficients  $N_{ab}^c$  in  $\mathcal{A}$  and the fusion coef-

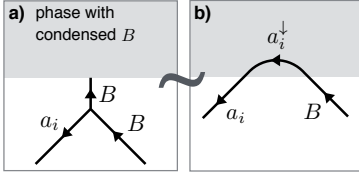


FIG. 1: Tunneling processes mediated by an anyon condensate. The gray region is a phase in which a boson  $B$  is condensed. a) Vertex of a boson  $B$  that localizes a zero mode of anyon  $a_i$ . In the condensed phase,  $B$  can be converted into an identity particle world line (not shown). By the axioms of anyon condensation, processes a) and b) are equivalent, i.e.,  $B$  can be converted into  $a_i$  by tunneling through the condensate.

ficients  $\tilde{N}_{rs}^t$  in  $\mathcal{T}$ . A corollary to Eq. (2) [25] is

$$d_a = \sum_{r \in \mathcal{T}} n_a^r d_r, \quad \forall a \in \mathcal{A}. \quad (3)$$

The restriction is compatible with conjugation to antiparticles, i.e.,  $n_a^t = n_{\bar{a}}^{\bar{t}}$ , where bar denotes the (unique) antiparticle of an anyon. We say particle  $a$  *condenses* if  $\varphi \in a^\downarrow$ , i.e.,  $n_a^\varphi \neq 0$ . Common knowledge in condensed matter physics says that any bosons can condense. However, it may also occur that a specific boson  $a$  cannot condense, i.e., there is no solution to the above equations with  $n_a^\varphi \neq 0$ . This is the situation we shall analyze in this paper.

Finally, the following definition is useful for formulating our no-go theorem: For a given boson  $b$ , a subset  $\mathcal{I}_b = \{a_1, \dots, a_m\}$  of anyons is called a *set of zero modes localized by  $b$*  [35] if for all  $i, j = 1, \dots, m$ :

1. The fusion products  $a_i \times a_j$  do not contain condensable bosons, except the identity if  $a_i = \bar{a}_j$ , [36]
2. all  $a_i$  are zero modes of  $b$ , by which we mean  $a_i \times b = b + \dots$ , (i.e.  $N_{a_i b}^b > 0$ )
3. if a particle  $a_i$  is in  $\mathcal{I}_b$  then so is its antiparticle.

Note that the choice of  $\mathcal{I}_b$  for a given boson  $b$  is not unique and that  $\mathcal{I}_b$  may or may not contain the identity. (The above conditions are satisfied in both cases.) Typically, we will be interested to find a set  $\mathcal{I}_b$  that is as large as possible. To motivate the terminology of the set  $\mathcal{I}_b$ , observe that  $N_{ab}^b > 0$  implies that  $a$  anyons can always be emitted or absorbed by  $b$ . Therefore,  $b$  must carry a zero-mode excitation of  $a$ . We can now state our first main result, a general condition under which a boson  $B$  cannot condense. It is an obstruction that is sufficient to show that condensation of  $B$  cannot occur.

*No-go theorem* — A boson  $B$  cannot condense if there exists a set  $\mathcal{I}_B$ , such that the sum of the quantum dimensions of all anyons in  $\mathcal{I}_B$  exceeds the quantum dimension of  $B$ , i.e., if

$$d_B < d_{a_1} + d_{a_2} + \dots + d_{a_m}. \quad (4)$$

*Proof.* We start by showing that all particles in  $\mathcal{I}_B$  do not split, and have distinct restrictions. This follows from inspection of Eq. (2) for  $t = \varphi$ ,  $a = a_i$ ,  $b = \bar{a}_j$ ,

$$\sum_{r \in \mathcal{T}} n_{a_i}^r n_{a_j}^r = \delta_{i,j} + \sum_{c \neq 1} N_{a_i \bar{a}_j}^c n_c^\varphi, \quad (5)$$

where we used  $n_c^\varphi = n_c^\varphi$ . By assumption, there are no condensable bosons in  $a_i \times \bar{a}_j$ , hence  $N_{a_i \bar{a}_j}^c$  and  $n_c^\varphi$  cannot be both nonzero for any  $c \neq 1$ . Thus  $\sum_r n_{a_i}^r n_{a_j}^r = 1$ , implying a single restriction  $a_i^\downarrow$  of  $a_i$ , with  $d_{a_i^\downarrow} = d_{a_i}$  using Eq. (3). Moreover,  $\sum_r n_{a_i}^r n_{a_j}^r = 0$  if  $i \neq j$ , implying that the restrictions of  $a_i \neq a_j$  are distinct particles.

With this knowledge about the restrictions of the  $a_i$ , Eq. (2) for  $t = \varphi$ ,  $a = a_i$ ,  $b = \bar{B}$  evaluates to

$$n_{\bar{B}}^{a_i^\downarrow} = n_B^{a_i^\downarrow} = \sum_c N_{a_i \bar{B}}^c n_c^\varphi \geq N_{a_i B}^B n_B^\varphi, \quad (6)$$

where we used  $N_{a_i \bar{B}}^{\bar{B}} = N_{a_i B}^B$ . Inserting this inequality in Eq. (3) for  $a = B$ , and using  $d_{a_i} = d_{a_i^\downarrow}$ , we have

$$d_B \geq n_B^\varphi \sum_{i=1}^m N_{a_i B}^B d_{a_i}. \quad (7)$$

It follows that in a situation where Eq. (4) holds, Eq. (7) implies  $n_B^\varphi = 0$ , i.e.,  $B$  does not condense. [Note that in the case  $N_{a_i B}^B > 1$ , a stronger form of Eq. (4) with  $d_{a_i}$  is replaced by  $N_{a_i B}^B d_{a_i}$  holds.]

To follow up with a pictorial representation of these equations, consider the tunneling of anyons across the domain wall as shown in Fig. 1, where each particle  $a$  in the uncondensed theory is converted into its restriction  $a^\downarrow$  in the gray region. Figure 1 (a) shows a vertex allowed by the fusion rule  $a_i \times B \rightarrow B$  in the uncondensed phase. The boson  $B$  enters the condensed phase, where it can disappear as it is part of the condensate (one of its restrictions is the vacuum  $\varphi$ , the world lines of which can be removed at will). By the fundamental assumption that fusion and condensation commute [which is at the heart of Eq. (2)], Fig. 1 (a) is equivalent to Fig. 1 (b). The latter represents a coherent tunneling process that is mediated by the condensate and converts  $B$  into any of the  $a_i$ . The existence of this process implies that the distinct restriction  $a_i^\downarrow$  of any  $a_i$  must be in the restriction of  $B$ . Hence, by Eq. (3), the quantum dimension of  $B$  must be large enough to accommodate all the distinct restrictions of the  $a_i$ , if  $B$  condenses. Therefore if we find sufficiently many  $a_i$  such that Eq. (4) holds,  $B$  cannot condense.  $\square$

Note that the no-go theorem does not a priori require knowing the braiding data of  $\mathcal{A}$  – although the modular tensor category structure fixes that data to some extent. The theorem involves only data obtainable from  $N_{ab}^c$ . We remark that the no-go theorem can only ever yield an obstruction against the condensation of non-Abelian bosons. For Abelian bosons, the theory after condensation can be constructed explicitly, which is a constructive proof that there is no obstruction. [25]

We now demonstrate that the no-go theorem is practically useful by considering three examples: (i) multiple layers of the Fibonacci TQFT, (ii) single layers of the  $\text{SO}(3)_k$  TQFT for  $k$  odd, and (iii) multiple layers of the latter. We will show that all these theories, while containing bosons, do not admit condensation transitions. All the bosons are noncondensable. Additional general results, concerning for instance TQFTs with a condensing Abelian sector and with only a single boson, are given in the Supplemental Information. [41]

*Example (i): Multiple layers of Fibonacci* — The Fibonacci category  $\mathcal{A}_{\text{Fib}}$  is a non-Abelian TQFT containing just one nontrivial particle  $\tau$  with a fusion rule  $\tau \times \tau = 1 + \tau$ , a topological spin  $\theta_\tau = e^{i4\pi/5}$ , and a quantum dimension  $d_\tau = \phi$  given by the golden ratio  $\phi = (1 + \sqrt{5})/2$ . As  $\mathcal{A}_{\text{Fib}}$  does not contain any nontrivial boson, it cannot undergo a condensation transition. We are interested whether the TQFT formed by  $N$  identical layers of  $\mathcal{A}_{\text{Fib}}$  i.e., the TQFT  $\mathcal{A}_{\text{Fib}}^{\otimes N}$ , admits a condensation transition. The TQFT  $\mathcal{A}_{\text{Fib}}^{\otimes N}$  contains  $2^N$  particles corresponding to all possible distributions of  $\tau$ -particles over the  $N$  layers. For each  $r = 0, \dots, N$  there are  $\binom{N}{r}$  so-called  $(r\tau)$  particles with  $\tau$ 's in exactly  $r$  layers, each with spin  $\theta_{(r\tau)} = e^{i4\pi r/5}$  and quantum dimension  $d_{(r\tau)} = \phi^r$ . The unique  $r = 0$  particle is the identity of  $\mathcal{A}_{\text{Fib}}^{\otimes N}$ . From the topological spin, the bosons in  $\mathcal{A}_{\text{Fib}}^{\otimes N}$  are  $(r\tau)$  particles with  $r = 5n$ ,  $n \in \mathbb{Z}$ . Using the no-go theorem, we show that none of these bosons can condense.

Using proof by induction on  $n \geq 1$ , we show that for any  $(5n\tau)$  boson  $B$ , there exists a set  $\mathcal{I}_{(5n\tau)}$  such that Eq. (4) holds. We first consider the case  $n = 1$ . Given a  $(5\tau)$  boson, we must construct a set  $\mathcal{I}_{(5\tau)}$  for this boson. Consider the set formed by all  $(2\tau)$  particles obtained by replacing any 3  $\tau$ 's in the boson with a 1. There are  $\binom{5}{2} = 10$  such  $(2\tau)$  particles for a given  $(5\tau)$  boson. They form a set  $\mathcal{I}_{(5\tau)}$  that obeys point 1–3 from the definition: point 1 holds as any product of two of these particles has at most 4  $\tau$ s and is therefore not a (potentially condensable) boson. Points 2 and 3 can be checked by using the Fibonacci fusion rules in each layer. Finally, Eq. (4) holds because

$$d_{(5\tau)} = \phi^5 < 10\phi^2 = \sum_{a_i \in \mathcal{I}_{(5\tau)}} d_{a_i} \quad (8)$$

evaluates to about  $11.1 < 26.2$ . We conclude that none of the  $(5\tau)$  bosons condense for any number  $N$  of layers of Fibonacci TQFT.

For the induction step, we assume that none of the  $(5n\tau)$  bosons can condense for  $n < n_0$ ,  $n_0 > 1$ , and we show that the same holds for the  $(5n_0\tau)$  bosons. Define  $r_0 := \lfloor (5n_0 - 1)/2 \rfloor$ , where  $\lfloor x \rfloor$  is the largest integer smaller than or equal to  $x$ . For a given  $(5n_0\tau)$  boson, form the set  $\mathcal{I}_{(5n_0\tau)}$  out of all  $(r_0\tau)$ -particles that are obtained by replacing any  $(5n_0 - r_0)\tau$ 's in the boson  $(5n_0\tau)$  with a 1. There are  $\binom{5n_0}{r_0}$  such  $(r_0\tau)$  particles. They form a set  $\mathcal{I}_{(5n_0\tau)}$  for  $(5n_0\tau)$ . In particular their fusion products can only contain  $(5n\tau)$ -bosons with  $n < n_0$ , which cannot condense by assumption. Equation (4) reads

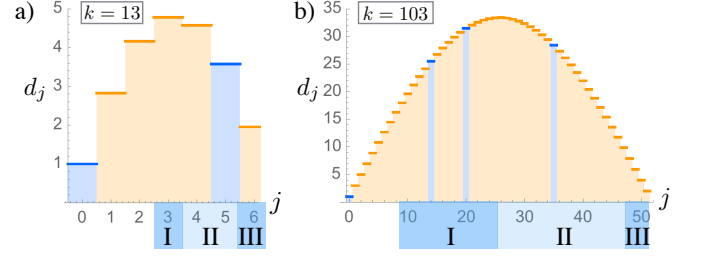


FIG. 2: Quantum dimensions and bosons (blue columns) for  $\text{SO}(3)_k$  theories with a)  $k = 13$  and b)  $k = 103$ . These are the smallest  $k$ , for which  $\text{SO}(3)_k$  contains two and four bosons, respectively. Indicated are also the ranges I–III defined in Eq. (13). The maximum quantum dimension coincides with the boundary between range I and II in Eq. (13). For instance, to apply the no-go theorem to the  $j = 5$  boson in a), choose  $\mathcal{I}_{j=5} = \{j = 2\}$  and use that  $d_5 \approx 3.6$  is smaller than  $d_2 \approx 4.2$ .

for this case

$$\phi^{5n_0} < \binom{5n_0}{r_0} \phi^{5n_0 - r_0}. \quad (9)$$

Using that  $r_0 \sim 5n_0/2$  and  $\binom{5n_0}{5n_0/2} \sim 4^{5n_0/2} / \sqrt{\pi 5n_0/2}$  for large  $n_0$ , we obtain that the right-hand side of Eq. (9) grows like  $4^{5n_0/2} \phi^{5n_0/2} / \sqrt{n_0}$ , asymptotically dominating the left-hand side. An explicit evaluation yields that Eq. (9) holds for any  $n_0 \geq 1$  in fact. We have thus shown that none of the  $(5n_0\tau)$  bosons can condense. This concludes the induction step and the proof that no boson in  $\mathcal{A}_{\text{Fib}}^{\otimes N}$  can condense.

*Example (ii): Single layer of  $\text{SO}(3)_k$*  — Our second example focuses on the (single-layer) TQFTs associated with the Lie group  $\text{SO}(3)$  at values of odd level  $k$ . They contain bosons for an infinite subset of  $k$ . We show that none of these bosons can condense. The  $\text{SO}(3)_k$  TQFTs with  $k$  odd have  $(k+1)/2$  anyons  $j = 0, \dots, (k-1)/2$  with

$$d_j = \frac{\sin\left(\pi \frac{2j+1}{k+2}\right)}{\sin\left[\pi/(k+2)\right]}, \quad \theta_j = e^{2\pi i j \frac{j+1}{k+2}}. \quad (10)$$

We note that for  $k$  odd, all particles have distinct quantum dimensions. The fusion rules are

$$N_{j_1 j_2}^{j_3} = \begin{cases} 1 & |j_1 - j_2| \leq j_3 \leq \min\{j_1 + j_2, k - j_1 - j_2\} \\ 0 & \text{else} \end{cases}. \quad (11)$$

The smallest odd  $k$  for which  $\text{SO}(3)_k$  contains a boson is  $k = 13$ , in which  $j = 5$  is a boson – an uncondensable one, as we shall see.

The topological spins  $\theta_j$  yield the condition  $j(j+1) = k+2$  for the lowest  $j$  that may correspond to a boson (aside from the vacuum  $j = 0$ ). (Frequently, this condition cannot be met with integer  $j$ , as in the  $k = 13$  example, and the lowest boson appears at even higher  $j$ .) We conclude that the first boson after  $j = 0$  cannot occur for  $j$  lower than

$$j_0 = \left\lceil \sqrt{k+9/4} - 1/2 \right\rceil. \quad (12)$$

We will now discuss separately bosons  $j$  in the three ranges (see Fig. 2 for two examples)

$$\text{I.} \quad j_0 \leq j \leq \lfloor k/4 \rfloor \quad (13a)$$

$$\text{II.} \quad \lfloor k/4 \rfloor < j \leq \frac{k-1}{2} - \left\lfloor \frac{j_0-1}{2} \right\rfloor \quad (13b)$$

$$\text{III.} \quad \frac{k-1}{2} - \left\lfloor \frac{j_0-1}{2} \right\rfloor < j \leq \frac{k-1}{2}. \quad (13c)$$

Due to Eq. (12), bosons  $j_B$  in range III have no bosons in their fusion product  $j_B \times j_B$ , other than the identity. Thus, from Eq. (2) for  $t = \varphi$ , and the fact that  $B$  are their own antiparticles, we conclude that they cannot split. Using Eq. (3) and the fact that they have  $d_{j_B} > 1$ , we conclude that they cannot restrict to the vacuum i.e., they cannot condense.

We now use our no-go theorem to show that bosons  $j_B$  in range I are non-condensable. Specifically, we show that the particles  $0 < j < \lfloor j_B/2 \rfloor$  form a set  $\mathcal{I}_{j_B}$  of  $j_B$  obeying Eq. (4). Before establishing that they satisfy the conditions for a set  $\mathcal{I}_{j_B}$ , let us show that Eq. (4) holds for  $\mathcal{I}_{j_B}$ . For large  $k$ , we can rely on the following asymptotic estimate. Using that the sine function in Eq. (10) is monotonously increasing with negative second derivative for  $j \leq \lfloor k/4 \rfloor$ , the estimate

$$2j_B + 1 < \sum_{j=1}^{\lfloor j_B/2 \rfloor - 1} (2j + 1) \quad (14)$$

implies Eq. (4) for  $j_B$  in range I. This inequality holds for all  $j_B \geq 10$ . Using Eq. (12) we conclude that it applies to all bosons in range I for  $k \geq 109$ . We verified explicitly that inequality (4) holds (using the exact values of the quantum dimensions) for all bosons in range I for  $k < 109$ . Finally, it is readily verified using Eq. (11) that  $\mathcal{I}_{j_B}$  form a set of zero modes localized by  $j_B$  provided that all bosons with  $j < j_B$  cannot condense. The proof then proceeds straightforwardly by induction.

We apply our no-go theorem successively to bosons  $j_B$  in range II in order of increasing  $j_B$ . Using the result that all bosons in range I are uncondensable, one verifies that the particles  $j$  with  $1 \leq j \leq \min\{k - 2j_B, \lfloor j_B/2 \rfloor - 1\}$  form a set  $\mathcal{I}_{j_B}$ . As for range I, we can estimate the quantum dimensions. From the relation  $\sin[\pi(2j_B + 1)/(k + 2)] = \sin[\pi(k - 2j_B + 1)/(k + 2)]$  we can estimate the quantum dimension of  $j_B$  using  $\sin[\pi(2j_B + 1)/(k + 2)] < \pi(k - 2j_B + 1)/(k + 2)$ . The quantum dimensions of the anyons in  $\mathcal{I}_{j_B}$  are estimated as for range I with  $\sin[\pi(2j + 1)/(k + 2)] < \pi(2j + 1)/(k + 2)$ . Using these estimates we find that if

$$k - 2j_B + 1 < \sum_{j=1}^{\min\{k-2j_B, \lfloor j_B/2 \rfloor - 1\}} (2j + 1) \quad (15)$$

holds, Eq. (4) follows. In the case  $k - 2j_B < \lfloor j_B/2 \rfloor - 1$ , Eq. (15) reduces to  $1 < (k - 2j_B)^2 + (k - 2j_B)$ , which is true for all  $j_B$  in range II for all  $k$ . In the case  $k - 2j_B > \lfloor j_B/2 \rfloor - 1$ , Eq. (15) simplifies to  $k + 2 < 2j_B + (\lfloor j_B/2 \rfloor)^2$ ,

which holds for all  $j_B$  in range II if  $k \geq 37$ . We verified explicitly that Eq. (4) holds for all bosons in range II if  $k < 37$  (they appear in  $k = 13, 19, 31$ ). This concludes our proof that no condensation transition is possible in the  $\text{SO}(3)_k$  TQFT for any odd  $k$ .

We note that this result can be readily extended to  $\text{SU}(2)_k$  with  $k$  odd, since  $\text{SO}(3)_k$  is the projection of  $\text{SU}(2)_k$  to anyons with integer  $j$ . One simply includes the half-integer  $j$  anyons in the theory (none of which are bosons). The sets  $\mathcal{I}_b$  as defined above remain the same and so do all the quantum dimensions. Hence, we also showed the noncondensability of  $\text{SU}(2)_k$ , with  $k$  odd. This is consistent with the ADE classification of  $\text{SU}(2)_k$  [37]: There are no off-diagonal modular invariant partition functions for odd  $k$  in  $\text{SU}(2)_k$  [38]. Thus, the no-go theorem provides a proof of this fact that is complementary to the ADE classification.

*Example (iii): Multiple layers of  $\text{SO}(3)_k$*  — We can show that any number of layers of  $\text{SO}(3)_k$ , with  $k$  odd, does not contain condensable bosons. Fixing  $k$ , the proof proceeds again by induction. As induction base, we proof that all multi-layer anyons with a nontrivial particle in only a single layer (and the identity anyon in the other  $k - 1$  layers) cannot condense nor split. To show that, we can use the single-layer result from Example (ii). For the induction step, we assume that for a fixed  $k_0 < k$  all multi-layer anyons with nontrivial particles in  $l$  layers,  $1 \leq l \leq k_0$ , cannot condense and do not split. We can then show that the same holds for multilayer anyons with nontrivial particles in  $k_0 + 1$  layers, completing the induction. The details of this proof are given in [41].

*Summary* — We have presented a generally applicable no-go theorem against the condensation of a topological boson and illustrated it with several examples. The proof of our theorem uses mostly the fusion (as compared to the braiding) information of the TQFT. We showed a connection between our results and the ADE classification of  $\text{SU}(2)_k$  theories, indicating that the no-go theorem might be useful for the classification of modular invariant partition functions of conformal field theories more broadly. [25] It would be interesting to study, whether other obstructions against boson condensation exist or whether our no-go theorem actually constitutes a necessary condition. In all examples we know, noncondensability is captured by the no-go theorem.

The no-go theorem can not only be used to directly classify (symmetry enriched) TQFTs, but also symmetry protected topological phases without intrinsic topological order. The classification of the latter is often related to the former upon gauging the protecting symmetry. [39, 40]

Finally, we note that a particle physics analogy of the no-go theorem is that a particle cannot decay if its mass is smaller than the total mass of its potential products. In this picture desintegration amounts to condensation and mass to quantum dimension.

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## SUPPLEMENTAL INFORMATION

### Proof for Example (iii): Multiple layers of $\text{SO}(3)_k$

In this section, we show that no condensation is possible in the TQFT  $\text{SO}(3)_k^{\otimes N}$  comprised of  $N$  layers of  $\text{SO}(3)_k$  for any odd  $k$  and any integer  $N$ . The proof goes by induction. We denote the particles in  $\text{SO}(3)_k^{\otimes N}$  with a shorthand notation. An anyon that has the identity particle from  $\text{SO}(3)_k$  in all layers, except for the  $k_0$  layers  $i_1, i_2, \dots, i_{k_0}$ , is denoted by  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0}}\}$ . Here  $1 \leq j_{i_l} \leq (k-1)/2$  can stand for any anyon from  $\text{SO}(3)_k$  (except the identity 0), for all  $l = 1, \dots, k_0$ .

#### Induction base

First, consider particles  $\{j_i\}$  with just one nontrivial anyon in some layer  $i$ . This will serve as the induction base. By the no-go theorem and our proof in Example (ii), we know that no bosons of form  $\{j_i\}$  can condense. [Use the particles with only one nontrivial in that same layer  $i$  to build the set  $\mathcal{I}_{j_i}$  as elaborated for Example (ii).] As a corollary, the anyons  $\{j_i\}$  do not split: when fused with themselves no condensable boson appears in the fusion product, which prevents splitting by Eq. (2) from the main text for  $t = \varphi$ .

#### Induction step

We assume that for any  $1 \leq l \leq k_0$  all  $\{j_{i_1} j_{i_2} \dots j_{i_l}\}$

1. do not condense and
2. do not split.

We now show the induction step, namely that all particles with nontrivial anyons in  $(k_0 + 1)$  layers  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0+1}}\}$  neither condense nor split.

We begin by showing that  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0+1}}\}$  cannot condense. The particles  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0+1}}\}$  can be obtained by fusing a  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0}}\}$  with a  $\{j_{i_{k_0+1}}\}$ , where  $i_{k_0+1} \notin \{i_1, \dots, i_{k_0}\}$ . In this case, Eq. (2) from the main text reads for  $t = \varphi$

$$\tilde{N}_{\{j_{i_1} \dots j_{i_{k_0}}\}^\downarrow, \{j_{i_{k_0+1}}\}^\downarrow}^\varphi = n_{\{j_{i_1} j_{i_2} \dots j_{i_{k_0}} j_{i_{k_0+1}}\}}. \quad (16)$$

Now, because of the uniqueness of the antiparticle,  $\tilde{N}_{\{j_{i_1} \dots j_{i_{k_0}}\}^\downarrow, \{j_{i_{k_0+1}}\}^\downarrow}^\varphi$  can be either 0 or 1. If it was 1,  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0}}\}^\downarrow$  would be the antiparticle of  $\{j_{i_{k_0+1}}\}^\downarrow$ . Because all particles are their own antiparticles, this would imply  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0}}\}^\downarrow = \{j_{i_{k_0+1}}\}^\downarrow$ . However, this is not possible for  $k_0 > 1$ , because the associativity of fusion would then also imply that  $\{j_{i_1}\}^\downarrow$  is the antiparticle (and coinciding with)  $\{j_{i_2} \dots j_{i_{k_0}} j_{i_{k_0+1}}\}^\downarrow$ , i.e.,  $\{j_{i_1}\}^\downarrow = \{j_{i_2} \dots j_{i_{k_0}} j_{i_{k_0+1}}\}^\downarrow$ . Remembering that  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0}}\}$ ,  $\{j_{i_2} j_{i_3} \dots j_{i_{k_0+1}}\}$  do not

split, and equating the quantum dimensions of the particles for these two identifications we have

$$\begin{aligned} d_{j_{i_1}} d_{j_{i_2}} \dots d_{j_{i_{k_0}}} &= d_{j_{i_{k_0+1}}}, \\ d_{j_{i_1}} &= d_{j_{i_2}} \dots d_{j_{i_{k_0}}} d_{j_{i_{k_0+1}}}. \end{aligned} \quad (17)$$

For  $k_0 > 1$ , this contradicts the fact that all nontrivial particles in this theory have quantum dimensions  $d > 1$ . This rules out the possibility  $\tilde{N}_{\{j_{i_1} \dots j_{i_{k_0}}\}^\downarrow, \{j_{i_{k_0+1}}\}^\downarrow}^\varphi = 1$  and shows that  $\{j_{i_1} j_{i_2} \dots j_{i_{k_0+1}}\}$  does not condense for  $k_0 > 1$ .

The case  $k_0 = 1$  needs to be considered separately, as both lines in Eq. (17) are identical in this case, and therefore do not lead to a contradiction. Assume that  $\tilde{N}_{\{j_{i_1}\}^\downarrow, \{j_{i_2}\}^\downarrow}^\varphi = 1$ . In the case  $j_{i_1} \neq j_{i_2}$ , we can rely on the following argument to disprove this assumption: As all anyons in  $\text{SO}(3)_k$  with  $k$  odd have distinct quantum dimension, it follows that the two anyons  $\{j_{i_1}\}$  and  $\{j_{i_2}\}$  restrict to distinct particles and in particular  $\varphi \notin j_{i_1}^\downarrow \times j_{i_2}^\downarrow$  – with Eq. (2) from the main text this implies that  $\{j_{i_1} j_{i_2}\}$  neither splits nor condenses. In the case  $j_{i_1} = j_{i_2} \equiv j$ , define  $\hat{j} \equiv \{j_{i_1} j_{i_2}\}$ . We want to show that  $\hat{j}$  does not condense. As there are no fermions in  $\text{SO}(3)_k$  with  $k$  odd,  $\hat{j}$  can only be a boson if  $\theta_j = 1$ , i.e., if  $\{j_{i_1}\}$  and  $\{j_{i_2}\}$  are bosons. Our no-go theorem applies to all bosons  $\{j_{i_1}\}$  and  $\{j_{i_2}\}$  with zero mode sets  $\mathcal{I}_{\{j_{i_1}\}}$  and  $\mathcal{I}_{\{j_{i_2}\}}$ . We can then use the set  $\mathcal{I}_{\hat{j}} = \mathcal{I}_{\{j_{i_1}\}} \times \mathcal{I}_{\{j_{i_2}\}}$ , containing the fusion product of any particle in  $\mathcal{I}_{\{j_{i_1}\}}$  with any particle in  $\mathcal{I}_{\{j_{i_2}\}}$ , to prove that  $\hat{j}$  cannot condense. To show that  $\mathcal{I}_{\hat{j}}$  is a set of zero modes of  $\hat{j}$ , the main challenge is to show that the product of any two elements from  $\mathcal{I}_{\hat{j}}$  cannot condense. The product of any two elements from  $\mathcal{I}_{\hat{j}}$  is always of the form  $\{j_{i_1} j_{i_2}\}$ . We have shown that when  $j_{i_1} \neq j_{i_2}$  such particles cannot condense. We therefore need only show that non-trivial particles of form  $\{j_{i_1} j_{i_2}\}$  with  $j_{i_1} = j_{i_2}$  both bosons cannot condense. In order to show they are not condensable, we can use the proof given for Example (ii). For that, observe that the anyons  $\hat{j}$  have the same fusion coefficients among themselves as the  $j$  anyons in  $\text{SO}(3)_k$  in Example (ii) have, i.e.,  $N_{\hat{j}, \hat{j}}^{j''} = N_{j, j}^{j''}$ , where  $j, j', j'' \in \text{SO}(3)_k$ . Recall that conditions 1–3 from the definition of a set of zero modes only depend on the fusion coefficients  $N_{j, j'}^{j''}$  and the information, which particles are bosons. Hence, conditions 1–3 are satisfied for  $\mathcal{I}_{\hat{j}}$  whenever they are satisfied for  $\mathcal{I}_j$  in Example (ii). It remains to show that  $\mathcal{I}_{\hat{j}}$  is of large enough quantum dimension to satisfy the fundamental inequality Eq. (4) from the main text. For  $\hat{j}$ , Eq. (4) from the main text takes the form

$$d_{\hat{j}} = d_j^2 < \left( \sum_{a \in \mathcal{I}_{\hat{j}}} d_a \right)^2 = \sum_{\hat{a} \in \mathcal{I}_{\hat{j}}} d_{\hat{a}}. \quad (18)$$

Upon taking the square root, this is equivalent to Eqs. (14) and (15) from the main text, which were shown to hold in Example (ii). Therefore the  $\hat{j} = \{j_{i_1} j_{i_2}\}$  anyons do not condense and all  $\{j_i\}$  have distinct restrictions.

We conclude that for any  $k_0 \geq 1$  only  $\tilde{N}_{\{j_{i_1} \dots j_{i_{k_0}}\}^\downarrow, \{j_{i_{k_0+1}}\}^\downarrow}^\varphi = 0$  is permitted and hence Eq. (16)

implies that  $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$  does not restrict to the identity  $\varphi$ , i.e., it does not condense. This proves the assumption 1 of the induction step for  $k_0 + 1$ .

To complete the induction step, we need to show that  $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$  does not split. For that, consider Eq. (2) from the main text for  $\{j_{i_1} \cdots j_{i_{k_0+1}}\}$  with itself and  $t = \varphi$

$$\sum_r \left( n_{\{j_{i_1} \cdots j_{i_{k_0+1}}\}}^r \right)^2 = \sum_c N_{\{j_{i_1} \cdots j_{i_{k_0+1}}\}, \{j_{i_1} \cdots j_{i_{k_0+1}}\}}^c n_c^\varphi = n_1^\varphi = 1. \quad (19)$$

We have used that none of the  $\{j_{i_1} j_{i_2} \cdots j_{i_l}\}$  with  $1 \leq l \leq k_0 + 1$  can restrict to the identity  $\varphi$  since they cannot condense. This implies none of  $\{j_{i_1} \cdots j_{i_{k_0+1}}\}$  splits, which proves the assumption 2 of the induction step for  $k_0 + 1$ .

We have thus shown inductively that none of the particles (except for the vacuum) restricts to the vacuum in the  $N$ -layer theory  $\text{SO}(3)_k^{\otimes N}$ . Thus, there is no condensate and with it no condensation in any number  $N$  of layers of  $\text{SO}(3)_k$  with  $k$  odd.

#### No-go theorem with Abelian sector

We have seen from the examples discussed in the main text, that the no-go theorem can often be used to not only show that individual bosons in a TQFT cannot condense, but that an entire TQFT is not condensable. Here, we extend this discussion to examples of TQFTs that have noncondensable substructures. This problem is motivated by physical examples: in the fractional quantum Hall effect, for instance, one frequently discusses phases that are described by a direct (or semi-direct) product of an Abelian and a non-Abelian TQFT. A simple example is the  $\mathbb{Z}_3$  Read-Rezayi state of bosons, which is described by the TQFT  $\mathcal{A}_{\text{Fib}} \times \mathbb{Z}_2$ . While such a theory admits condensations, already in the  $\mathbb{Z}_2^{\otimes N}$  sector, when enough layers are considered, one has the intuition that the noncondensability of Fibonacci should still constrain the possible condensations.

**Lemma 1.** *Consider a TQFT*

$$\mathcal{A} \times \mathcal{X}, \quad (20)$$

where  $\mathcal{X}$  is an Abelian TQFT (i.e., all its anyons have quantum dimension 1). Further, for all particles  $b \in \mathcal{A}$  (not only for the bosons), except for the vacuum, let there exist a set  $\mathcal{I}_b = \{a_1, \dots, a_m\}$  of zero modes of  $b$ , containing anyons from  $\mathcal{A}$ , such that the quantum dimensions satisfy

$$d_b \leq \sum_{i=1}^m d_{a_i}. \quad (21)$$

Then, any possible condensation transition will lead to a theory of the form

$$\mathcal{A} \times \mathcal{Y}, \quad (22)$$

where the Abelian TQFT  $\mathcal{Y}$  can be obtained from  $\mathcal{X}$  through a condensation.

*Proof.* This Lemma follows almost directly from the no-go theorem. Let us denote a particle from  $\mathcal{A} \times \mathcal{X}$  by the pair  $(b, x)$  where  $b \in \mathcal{A}$  and  $x \in \mathcal{X}$ . If  $(b, x)$  is boson, we can show that it has to be an uncondensable one, except if  $b = 1$ . The set

$$\mathcal{I}_{(b,x)} = \{(a_1, x), \dots, (a_m, x)\}, \quad (23)$$

(where  $a_1, \dots, a_m$  form a set  $\mathcal{I}_b$  of zero modes of  $b$  whose existence is guaranteed by assumption) satisfies all the conditions 1–3 form the definition of a set of  $(b, x)$  zero modes. Since  $x$  is an Abelian particle,  $d_x = 1$  and Eq. (21) directly implies that the sum of the quantum dimensions of the particles in  $\mathcal{I}_{(b,x)}$  satisfies the inequality (4) from the main text. Hence,  $(b, x)$  cannot condense. In turn, this implies any condensable boson in  $\mathcal{A} \times \mathcal{X}$  is of the form  $(1, x)$ . A condensate of this form is transparent to the anyons in  $\mathcal{A}$  and will thus leave this sub-TQFT unaffected. It will only induce a condensation  $\mathcal{X} \rightarrow \mathcal{Y}$ , so that the final theory is of the form (22).  $\square$

We return to the example of  $\mathcal{A}_{\text{Fib}} \times \mathbb{Z}_2$ . Consider  $N$  layers of this theory, i.e.,  $\mathcal{A}_{\text{Fib}}^{\otimes N} \times \mathbb{Z}_2^{\otimes N}$ . This multi-layer TQFT satisfies all assumptions of Lemma 1: for each anyon  $b \in \mathcal{A}_{\text{Fib}}^{\otimes N}$ , a choice for the set  $\mathcal{I}_b$  is given by  $\mathcal{I}_b = \{1, b\}$ . This is so because all possible bosons appearing in the fusion product of  $b \times b$  are uncondensable by the no-go theorem and the sum of the quantum dimensions of  $\mathcal{I}_b$ , given by  $1 + d_b$  is larger than  $d_b$ . We conclude that the  $\mathcal{A}_{\text{Fib}}^{\otimes N}$  structure is preserved under any condensation transition in such a theory.

#### General constraints on boson condensation

In this section, we list lemmas that pose other general constraints on condensation transitions in TQFTs.

**Lemma 2.** *Suppose  $S = \{a_1, \dots, a_m\}$  is a collection of particles in a TQFT  $\mathcal{A}$  with  $a_i \times \bar{a}_i$  containing no bosons other than the identity – i.e.,  $n_{a_i}^t = \delta_{a_i}^t$  and  $a_i$  does not split. Moreover assume  $a_i^\dagger \neq a_j^\dagger$  for  $i \neq j$ . Then if a boson  $B$  appears in the fusion of  $a_i$  and  $\bar{a}_j$ ,  $a_i \times \bar{a}_j = B + \dots$  for any  $i \neq j$ , then  $B$  is not condensable.*

*Proof.* Using Eq. (2) from the main text for  $a = a_i$ ,  $b = \bar{a}_j$  and  $t = \varphi$ , we have  $\delta_{ij} = \sum_t n_{a_i}^t n_{\bar{a}_j}^t = \sum_c n_c^\varphi N_{a_i \bar{a}_j}^c$ . For  $i \neq j$  we get  $\sum_B n_B^\varphi N_{a_i \bar{a}_j}^B = 0$ . So if boson  $B$  appears in  $a_i \times \bar{a}_j$ , we must have  $n_B^\varphi = 0$ , so that  $B$  is not condensable.  $\square$

**Lemma 3.** *Consider a TQFT  $\mathcal{A}$  with no fusion multiplicity and just one boson  $B$ . If  $B$  is condensed then either  $B$  is abelian or  $B^\dagger = \varphi + r$  where  $r$  is a single anyon.*

*Proof.* As there is just a single boson,  $B = \bar{B}$ . Equation (2) from the main text implies

$$\sum_t n_B^t n_B^t = \sum_c n_c^\varphi N_{BB}^c = 1 + n_B^\varphi N_{BB}^B. \quad (24)$$

Notice, however, that the left-hand side is greater or equal to  $n_B^\varphi n_B^\varphi$ . For condensation, this implies  $n_B^\varphi = 1$ , and tells us that  $\sum_t n_B^t n_B^t = 1$  or 2. In the former case,  $B^\downarrow = \varphi$ . This implies  $d_B = 1$ , and so  $B$  is a quantum dimension 1 boson hence must have  $N_{BB}^a = \delta_{a,1}$ . In the latter case,  $B$  restricts to just two particles with multiplicity 1 each, so that  $B^\downarrow = \varphi + r$ .  $\square$

**Lemma 4.** *With the conditions of Lemma 3, and assuming  $B$  has  $d_B > 1$ , condensation of  $B$  can only occur if  $N_{BB}^a N_{BB}^b \leq N_{ab}^B$  for all anyons  $a$  and  $b$  of  $\mathcal{A}$ .*

*Proof.* Lemma 3 shows  $B^\downarrow = \varphi + r$ , where  $r$  a simple object. Consider  $a \in \mathcal{A}$  where  $a \neq 1, B$ . Equation (2) from the main text for  $b = B$  and  $t = \varphi$  reads

$$\begin{aligned} n_a^r n_B^r &= n_a^r \\ &= N_{aB}^B n_B^\varphi \\ &= N_{aB}^B \end{aligned} \quad (25)$$

Consider now Eq. (2) from the main text for  $b \neq 1, B$  and for  $a \neq b$ , which gives

$$\sum_t n_a^t n_b^t = N_{ab}^B \geq n_a^r n_b^r \quad (26)$$

Combining the  $a, B$  and  $b, B$  and  $a, b$  equations gives the inequality

$$N_{BB}^b N_{BB}^a \leq N_{ab}^B. \quad (27)$$

$\square$